

What was wrong in Michell's paper of 1904?

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Abstract The Writers of this paper published a discussion (Vázquez Espí and Cervera Bravo, 2011) on a paper by Sokół and Lewiński (2010). The discussion was replied (Sokół and Lewiński, 2011). Afterwards Rozvany (2011) has written a Discussion with comments on this exchange. Several Rozvany's comments have to do with "an error in Michell's (1904) paper". The Writers analyse herein Rozvany's statements about such an error.

Keywords peer-review method · classic papers' reading · Maxwell's problems

1 Introduction

Rozvany (2011) argues against Writers' arguments with two main points: (i) their notions "are controversial and difficult to understand"; and (ii) "it is difficult to see, why [the Writers] try to convert the Michell problem solved by Sokół and Lewiński (2010) in Maxwell problems, since the latter are a subclass of the former". The Writers have given a fully detailed explanation of their position about these two points in Vázquez Espí and Cervera Bravo (2012a).

The main concern herein has to do with the following statements by Rozvany (2011): "[The Writers] are certainly right on the importance of statically determinate support conditions for Maxwell structures." (It

should be stressed that the Writers did not use this expression but "*all external forces are known*", Vázquez Espí and Cervera Bravo 2011:724). Rozvany adds that he "pointed out (Rozvany, 1996) a serious error in Michell's proof, in ignoring the fact that Maxwell has this restriction on the considered class of problems".

Writers' hypothesis is that Rozvany has made two main errors interpreting Michell's paper and as a consequence he concludes that Michell was seriously wrong. (Note that Rozvany's paper of 1996 dealt with *shortcomings*, or *restrictions* in Michell's theory; the use of the *error*'s term is recent). The Writers will present a concise proof of their hypothesis in the next two sections.

2 Michell's criterion

Rozvany (2011:Eq. (1)) says that:

"Michell has derived optimality criteria for *least weight* trusses with a *stress constraint* and a single *load* condition. It involves a *strain field* (currently termed 'adjoint strain field') with the following properties: $\bar{\varepsilon} = k \operatorname{sgn} F$ (for $F \neq 0$), $|\bar{\varepsilon}| \leq k$ (for $F = 0$), $k = 1/\sigma_P$, where $\bar{\varepsilon}$ is the adjoint strain, F is the member force, k is a constant, and σ_P the permissible stress".

This description is not accurate when compared with Michell's text. Since some differences are very subtle, it is necessary to recall the latter.

Michell (1904) did not use the word "weight" and definitively not the expression "minimum weight". The reason is he was deriving a criterion for trusses of minimal "quantity $\sum |F|\ell$ " (p. 590), ℓ being the member length. This is the quantity that the Writers name "quantity of structure" or "stress volume" \mathcal{Q} (Vázquez Espí

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and Cervera Bravo, 2011, 2012a). Michell’s aim is justified by his Eqs. (1), (2) and (3) (p. 589–590) that completely define the class of structural problems he is dealing with, for which the Maxwell’s assertion on constancy of C in Eq (1) —our \mathcal{M} in (Vázquez Espí and Cervera Bravo, 2011), closely related with \mathcal{Q} — is plenty of usefulness.

Michell does not mention “load” but “applied forces” or “given forces”, and within the context of Maxwell’s work cited by Michell, those forces must be in equilibrium. Note that *given external forces in equilibrium* condition is not logically equivalent to “statically determinate support” one. A statically *undetermined* support condition can be tackled with the former providing that a set of external forces in equilibrium is selected from the infinitely many sets that are compatible with the given support condition.

Michell considers distinct stress levels of tension and compression, P and Q (Eq. (2) and (3), p. 589–590). Furthermore, according to Maxwell’s lemma and Michell’s lemma —i.e., Eqs. (1) and (3)—, P and Q have no influence in the optimal value of \mathcal{Q} , providing that both of them are strictly positive and finite. Hence to be respectful to Michell’s text, the equation $k = 1/\sigma_P$ —which is not from Michell— should be interpreted with σ_P being dependent on the stress sign, i.e., not equal in tension or compression, thus resulting in Hemp’s optimality criteria (see Rozvany, 1996) if Prager-Shield’s condition is used —as Rozvany does, see Rozvany and Sokół 2012: Eq. (1)—, neither in Michell’s, nor in Rozvany’s unique k .

Michell’s criterion was derived from two different theorems—however this pair is commonly known as “Michell’s theorem”.

Let \mathcal{D} be the set of displacement fields such that $\forall d \in \mathcal{D}$, $|\varepsilon^d| \leq \Delta_d$ for all points and directions of the considered domain Ω , Δ_d being a finite, strictly positive deformation, e.g., Rozvany’s *unique k* in his rewriting of Michell’s.¹ Let \mathcal{S} be the set of trusses such that they are enclosed into Ω and that are feasible for the given set of external forces in equilibrium, according to Michell’s Eq. (1).

First Michell’s theorem (p. 590) states that $\forall A \in \mathcal{S}$ and $\forall d \in \mathcal{D}$:

$$\mathcal{Q}(A) \geq \frac{W^d}{\Delta_d} \quad (1)$$

¹ Working with a finite “ground structure”—or a “basic truss”—, Rozvany’s statement “ $|\varepsilon| \leq k$ (for $F = 0$)” can suggest that only the inactive members of the ground structure in the optimal truss have to fulfill this condition, which is a weaker condition than Michell’s bound in displacement fields, as the latter is to be applied to all points and directions of Ω .

W^d being the virtual work of the given external forces when the domain Ω undergoes the displacement field d . Hence:

$$\mathcal{L} = \sup_d \left\{ \frac{W^d}{\Delta_d} : d \in \mathcal{D} \right\} \leq \inf_A \{ \mathcal{Q}(A) : A \in \mathcal{S} \} \quad (2)$$

\mathcal{L} is a lower bound of \mathcal{Q} and it was named “the limit of economy of material” by Michell (1904:591). Also note that any truss cost defined in the same format as the geometrical volume is —i.e., Michell’s Eq.(2)— will be minimal for any truss with minimal \mathcal{Q} .

Second Michell’s theorem (p. 590–591) states that if a couple (T, M) exists for Ω , such that $T \in \mathcal{D}$, $M \in \mathcal{S}$, and $\varepsilon_i^T F_i^M = \Delta_T |F_i^M|$ for every truss member i then the truss M “attains the limit of economy of material” in Ω , $\mathcal{Q}(M)$ “is a minimum, and consequently from (3) the volume of material in the frame M is also a minimum.” (p. 591).

So that Michell’s criterion requires:²

- (i) the existence of a couple (T, M) in Ω ,
- (ii) a finite bound strictly positive Δ_T on T , and
- (iii) the conditions $\varepsilon_i^T F_i^M = \Delta_T |F_i^M|$ on all the members of M .

According to these remarks, it is difficult to understand that Rozvany (2011) could say that Michell was wrong “ignoring the fact that Maxwell has” the restriction to “statically determinate support conditions” “on the considered class of problems”. It is true that all examples that Michell presents can be described as having

² It should be noted that as Michell does not show any proof of existence of a couple (T, M) for every set of given external forces in equilibrium, Michell’s criterion is to be considered as a sufficient condition only. In fact, Michell defines three class of frames which can fulfil the criterion: a special class with “all the bars with stresses of the same sign” (p. 591) and two general classes: “tangents and involutes” and “orthogonal systems of equiangular spirals” (p. 593), according to his declared aim: “Starting from this result [Maxwell’s lemma], we can find in *certain cases* lower limits to the quantity of material necessary to sustain given forces, and also assign the forms of frames which attain the limit of economy” (p. 589). In spite of a sustained research effort on this subject, it is not proved that Michell’s criterion is also necessary. The best result the Writers know up to date is that although a maximizer/minimizer pair for (2) always exists (Bouchitté et al, 2008: eqs. (2.22) and (2.24), and Proposition 2.1), the minimizer of the right-hand side of inequation (2) “may not be a Michell truss” (Gangbo, 2012). Bouchitté et al (2008: eq. (2.25) and theorem 3.1) have formulated a new problem which is a relaxation of Michell’s original problem and they have proved that both infima are equal. Moreover, “if we could prove [the] existence of a minimizer in [eq.] (3.6) [*ibidem*], we will use the optimum [Radon measure] γ to construct a minimizer σ^o which we know will be a Michell truss.” (Gangbo, 2012). Bouchitté et al (2008:1601–1602, Problem 5.1) conclude saying: “We strongly believe that our approach could be a useful tool to investigate the properties of optimal structures. However, it is still necessary to prove the existence of a minimizing measure [for the new problem]”.

this restriction. But note that Michell *starts* his paper with Maxwell's lemma and outlining the constancy of Maxwell's number \mathcal{M} . It implies that a set of external forces in equilibrium is given, so in any case these forces must be completely determined—either statically or *by any other method*, e.g. designer's decision. Moreover, no displacement or support conditions are found in Michell's work. Perhaps Cox (1965) found inspiration in these facts when he made a clear difference between “Free loading” and “Fixed boundary” classes of problems, the former being the class of Maxwell's problems, and the latter the class with any kind of displacement constraints. (Note that the intersection of these two sets is the set of problems with “statically determinate support conditions”, being this a proper subset of them both.)

3 Rozvany's error in a paper of 1996

In the paper “Some shortcomings in Michell's truss theory”, Rozvany (1996) looks for an explanation of the discrepancy between Michell's and Hemp's criteria. He reproduces exactly Michell's Eq. (2), speaking of “*any* statically determinate truss” (p. 244). But it should be stressed again that Michell always works with *given external forces in equilibrium*. Rozvany (1996:244) adds that “the volume [of the truss] can also be calculated from the «dual formula»,”

$$V = \frac{1}{2} \left(\frac{1}{P} + \frac{1}{Q} \right) \cdot W^T \quad (3)$$

“if we take $k = 1$ ”. And *this is wrong*. Rozvany continues saying: “Examples of these volume equations are given by Michell. . .”. And this is true except for Michell's example 3 (1904: Fig. 4, p. 596–597), which corresponding equation does not fulfill (3).

The correct “dual formulation” was well established for many authors (e.g., Cox 1965:87, Eq.(121)) as:

$$V = \frac{1}{2} \left\{ \left(\frac{1}{P} + \frac{1}{Q} \right) \mathcal{Q} + \left(\frac{1}{P} - \frac{1}{Q} \right) \mathcal{M} \right\} \quad (4)$$

and for a truss fulfilling Michell's criterion—if it exists:

$$V = \frac{1}{2} \left\{ \left(\frac{1}{P} + \frac{1}{Q} \right) \frac{W^T}{\Delta_T} + \left(\frac{1}{P} - \frac{1}{Q} \right) \mathcal{M} \right\} \quad (5)$$

Note that, with $\Delta_T = 1$, if either $\mathcal{M} = 0$ or $P = Q$, the last equation leads to Rozvany's wrong “dual formula”, (3) herein.

Rozvany (1996) tries in section 3 an “illustrative example” however it was rather unfortunate. The example has displacement constraints (a “Fixed boundary” problem) so Michell's theory is not applicable (Maxwell's

number varies with solutions, i.e., Michell's *start* is not fulfilled). Moreover, Rozvany uses (3) with unequal allowable stresses, but by coincidence, the Maxwell number of the conjectured Michell truss was null in the selected example—Rozvany (1996: Fig. 1b), from Prager and Rozvany (1977: Fig. 1)—, so he had no chance to detect any discrepancy between his primal and wrong dual formulae. (See a detailed analysis of this example in Vázquez Espí and Cervera Bravo 2012b:4–5.)

Therefore the “critical examination of Mitchell's proof” in section 4 makes no sense as it is based on all these misunderstandings and wrong outcome of the unique example that Rozvany considers.

4 Conclusion

The Writers conclude giving an answer to the title question: *nothing is wrong in Michell's paper*—or at least, the errors pointed out by Rozvany do not exist.

One of the Writers warned time ago (see Cervera Bravo, 1982: note 90) against the careless rewrite of old texts, as the differences between conceptual contexts can seriously distort their original meaning, thus diffculting the trace of conceptual or methodological progress.

Perhaps the main moral of this story is that the peer-review method—that was so useful during XVIII and XIX centuries with Royal Society's format—performs somehow bad nowadays with its current characteristics. Finding a new method that will perform well for trapping normal, human errors is pressing.

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